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The Border Model in One Dimension

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The magnetic susceptibility is studied by the methods of series analysis for the one-dimensional border model (a special case of the continuous-spin Ising model). The structure of this model is analyzed and two conjugate pairs of singularities are found near the real (physical) temperature axis. All the numerical results are consistent with the previously known rigorous results, but do not add to the knowledge of the critical properties.

KEY WORDS: Ising model; border model; Padé approximants; series analysis; universality; pluralism; field theory.

1. INTRODUCTION AND SUMMARY

Although the method of using perturbation series in the study of critical phenomena was not originated by Domb (see, for example, Fuchs,[1] Yvon,[2,3] and Rushbrooke and Scoins[4]), it is greatly to his credit that he recognized the value of this approach and contributed strongly to it (Domb,[5] Domb and Sykes[6]). In addition, he built up at King's College London a center of world expertise in this area. In the years since then, this area has considerable grown and developed. The present work does follow in the line from those developments in that we consider by means of series analysis the question of the critical parameters of the one-dimensional border model. In addition, we show how an approach, interrelated with a model's scaling limit field theory, which was previously used for investigating universality and pluralism in higher dimensions can be extended to the case of one dimension with its special feature of a zero-temperature critical temperature.

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In Section 2 we introduce the border model and show how the rigorous results available for the critical behavior of the correlation length can be used to bound the critical behavior of the magnetic susceptibility. In Section 3 we explore the relation between the unique field theory corresponding to the critical behavior of the one-dimensional, continuous-spin Ising model, and the critical exponents. It is found, since this model exhibits plural universality classes, that the popular method for extracting the critical behavior from the field theory alone must fail in this case. In Section 4 we examine the results for the magnetic susceptibility that can be deduced from the high-temperature series expansions. We find that these results are compatible with the known rigorous results, but that more series coefficients are required to have a hope of a reasonable estimate of the unknown critical parameters.

2. ONE-DIMENSIONAL BORDER MODEL

We will study the one-dimensional border model.[7] It is a special case of the continuous-spin Ising model. The partition function for the continuous-spin model is

$$Z = M^{-1} \int_{-\infty}^{+\infty} \cdots \int \prod_{i=1}^{N} \exp\left[\sum_i (K s_i s_{i+1} - \tilde{A} s_i^2 - \tilde{g}_0 s_i^4 + H s_i)\right] \tag{2.1}$$

where M is a formal normalization constant such that $Z(K = H = 0) = 1$, N is the number of lattice sites, $K = J/kT$, with J the exchange integral, k is Boltzmann's constant, and T is the absolute temperature. The condition $\langle s_i^2 \rangle = 1$ when $K = H = 0$ determines $\tilde{A}$ as a function of $\tilde{g}_0$. The border model is defined by the further restriction that $\tilde{A} = 0$, which corresponds to $\tilde{g}_0 = \tilde{g}_b = [\Gamma(3/4)/\Gamma(1/4)]^2$.

In a recent paper Baker[8] has proven for the continuous-spin Ising model that as $T \to 0$, i.e., $K \to \infty$, the critical point of this model, one has

$$C_H \asymp \frac{kK^2}{2\tilde{g}_0} \tag{2.2}$$

where C_H is the specific heat per lattice site at constant magnetic field ($H = 0$), and

$$\ln \xi \asymp uK^2, \qquad \frac{1}{4\tilde{g}_0} \leqslant u \leqslant \frac{1}{\tilde{g}_0} \tag{2.3}$$

where ξ is the "true" correlation length. These results are in contrast to the usual spin-1/2 Ising model results

$$C_H \asymp kK^2 \operatorname{sech}^2(K), \qquad \xi \asymp \tfrac{1}{2}e^{2K}, \qquad \chi \asymp e^{2K} \tag{2.4}$$

We will study the (reduced) magnetic susceptibility χ. In one dimension, we may write it as

$$\chi = \sum_{-\infty}^{+\infty} \langle \sigma_0 \sigma_j \rangle = \langle \sigma_0^2 \rangle \left\{ 1 + 2 \sum_{j=1}^{\infty} \frac{\langle \sigma_0 \sigma_j \rangle}{\langle \sigma_0^2 \rangle} \right\} \tag{2.5}$$

which, by reflection positivity,[9] may be re-expressed as

$$\chi = \langle \sigma_0^2 \rangle \left\{ 1 + 2 \sum_{j=1}^{\infty} \int_{\xi^{-1}}^{\infty} d\mu(t)\, e^{-tj} \right\}, \qquad d\mu \geqslant 0, \qquad \int d\mu(t) = 1 \tag{2.6}$$

Now in terms of the largest two eigenvalues of the transfer matrix λ_1 and λ_2, the correlation length is defined as $\xi = -1/\ln(\lambda_2/\lambda_1)$. Thus, as the integrand is positive and $\forall j$, e^{-tj} is monotonically decreasing in t, we may replace e^{-tj} in (2.6) by its value at the lower limit of the integral, so we conclude that

$$\chi \leqslant \langle \sigma_0^2 \rangle \left\{ 1 + 2 \sum_{j=1}^{\infty} \left(\frac{\lambda_2}{\lambda_1} \right)^j \right\} = \langle \sigma_0^2 \rangle \frac{1 + \lambda_2/\lambda_1}{1 - \lambda_2/\lambda_1} \tag{2.7}$$

Asymptotically as $K \to \infty$, i.e., as $\lambda_2/\lambda_1 \to 1$, we have

$$\chi \leqslant 2 \langle \sigma_0^2 \rangle \xi \tag{2.8}$$

which provides an asymptotic upper bound on the behavior of χ in terms of the results in (2.3).

3. RELATION OF THE CRITICAL EXPONENTS TO THE ONE-DIMENSIONAL FIELD THEORY

There does exist a well-defined and unique field theory for all $\tilde{g}_0 > 0$ for the critical point limit of the continuous-spin Ising model, as has been shown by Isaacson[10] and Marchesin.[11] The variation that we have observed with $\tilde{g}_0$ relates exclusively to the way in which this limit is approached. It stands to reason that it is not going to be possible for more than one set of critical behavior to be computed from a unique field theory. We will now see this point in some detail. The field theory approach to the calculation of the critical exponent ν relies[12] on the result

$$2 - \frac{1}{\nu} = 3 \lim_{g_0 \to \infty} g_0 \left. \frac{\partial \ln(Z_{(2)}/Z_3)}{\partial g_0} \right|_{\xi} = 0 \tag{3.1}$$

where $g_0 = 24\tilde{g}_0 a^3/K^2$, the $Z_{(2)}$ and Z_3 are renormalization constants, and a is the formal lattice spacing, which in the current case is just $a = 1/\xi$. The field theory approach in one dimension has been extensively studied by Nickel[13] at least insofar as the analysis of the Callan–Symanzik beta function, which determines the renormalized coupling constant, is concerned. There was no clear indication of a lack of universality in this analysis, which looked no different than that for higher dimensions. The one-dimensional analysis was much more extensive, however, and a confluent singularity was found at the critical point. This singularity was blamed for the very slow convergence of the coupling constant series expansion, which is known to 35 terms. Setting the difference between the Eq. (3.1) determination and the normal statistical mechanical one equal to zero, Baker[14] derived an equation for the critical amplitude of the correlation length under the assumption that $\xi \asymp D_+(\tilde{g}_0)(1 - K/K_c)^{-\nu}\, \forall \tilde{g}_0$. On the basis of this equation, he concluded for spatial dimension $(1 < d < 4)$ that it was inconsistent to assume that ν was the same for the border model, $\tilde{g}_0 = \tilde{g}_b$, as it is for $\tilde{g}_0$ nearby. As a sidelight, one might wonder in the original derivation whether the form (2.1) is "more natural" or whether the special point $\tilde{A} = 0$ should have been instead defined with the spin-spin coupling term a perfect square, $-\frac{1}{2}K(s_i - s_{i+1})^2$, so that $\tilde{A} + K$ would replace $\tilde{A}$ in determining the special point. In fact, if this change is made, there is no difference in the basic result (3.3) quoted below, because the shift is proportional to K. As was shown in ref. 8, for the one-dimensional continuous-spin model the appropriate asymptotic behavior is

$$\xi \asymp \exp[u(\tilde{g}_0)\, K^2 + v(\tilde{g}_0)\, K + w(g_0, K)], \qquad w = o(K), \qquad \text{as } K \to \infty \tag{3.2}$$

for $\tilde{g}_0 > 0$. Baker's[14] basic result [his Eq. (43)] is

$$\lim_{\substack{a \to 0 \\ \tilde{g}_0 \text{ fixed}}} a \frac{d}{da} \{\ln[2\tilde{A}(\tilde{g}_0)(K^{-1}(g_0, a) - K^{-1}(\tilde{g}_0, 0))\, a^{-2}]\}|_{g_0} = 0 \tag{3.3}$$

Note that the terms discarded in the derivation of (3.3) in ref. 14 are also negligible here because of the form of (3.2). In our case, the one-dimensional continuous-spin model with $\tilde{g}_0 > 0$, the term $K^{-1}(\tilde{g}_0, 0) = 0$ because the critical temperature is zero. If we solve (3.2) for K as a function of ξ and substitute it into (3.3), we get

$$\lim_{\substack{a \to 0 \\ \tilde{g}_0 \text{ fixed}}} a \frac{d}{da} \left\{ \ln \left[\frac{2\tilde{A}(g_0)}{a^2\{-v \pm [v^2 - 4u(w - \ln \xi)]^{1/2}\}/2u} \right] \right\} \Bigg|_{g_0} = 0 \tag{3.4}$$

which is, to leading order as $\xi \to \infty$,

$$\lim_{\substack{a \to 0 \\ \tilde{g}_0 \text{ fixed}}} a \frac{d}{da} \left\{ \ln \left[\frac{2\tilde{A}(g_0)}{a^2\{[(\ln \xi)/u]^{1/2} - v/2u\}} \right] \right\} \Bigg|_{g_0} = 0 \tag{3.5}$$

If we now perform the differentiation and take the limit as $\xi \to \infty$, (3.5) becomes

$$-2 - 3\tilde{g}_0 \left[\frac{\tilde{A}'(\tilde{g}_0)}{\tilde{A}(\tilde{g}_0)} + \frac{u'(g_0)}{2u(\tilde{g}_0)} \right] = 0 \tag{3.6}$$

Next if we integrate (3.6) with respect to $\tilde{g}_0$, we obtain

$$u(\tilde{g}_0) = \frac{\Theta}{\tilde{g}_0^{4/3} \hat{A}(\tilde{g}_0)^2} \tag{3.7}$$

where Θ is a constant independent of $\tilde{g}_0$. Now, it was shown in ref. 8 that

$$\tfrac{1}{4} \leqslant \tilde{g}_0 u(\tilde{g}_0) \leqslant 1 \tag{3.8}$$

which is clearly violated near $\tilde{g}_0 = 0$, $\tilde{g}_b$, and ∞ because $\tilde{A}(0) = 1/2$, $\tilde{A}(g_b) = 0$, and $\lim_{\tilde{g}_0 \to \infty} \tilde{A}(\tilde{g}_0)/\tilde{g}_0 = -2$. As we pointed out at the beginning of this section, it is not surprising that in the light of hindsight the field theory approach should fail here, and indeed it does. However, with only weaker information than the exact solution, this approach does correctly point to trouble with a universality hypothesis for all $\tilde{g}_0 > 0$.

4. SERIES ANALYSIS

In order to study the magnetic susceptibility, we use the series expansion of Nickel[15] of 21 terms, which he has kindly made available to us prior to publication. From (2.8) and (2.3) we expect that $\ln \chi$ will be bounded from above by a term proportional to K^2. Under reasonable hypotheses, it is not hard to show, since we know that $\ln \xi$ is definitely proportional to K^2 as $K \to \infty$, that $\ln \chi$ shares this property. Therefore we think it reasonable to study $\ln \chi$ by Padé approximants (see, e.g., ref. 16). In Table 1 we give some results. They are converged to the number of figures (with a possible error of one or two in the last figure) reported and are based on the $[N+2/N](K)$ Padé approximants, which have the correct asymptotic behavior at infinity. The reason that the convergence deteriorates where it does is straightforward enough to discover. The Padé approximants to $d \ln \chi/dK$ show that there is a pair of singular points at about $0.855 \pm 0.372i$ and another pair at about $-0.814 \pm 0.305i$. The natural cut structure for the Padé approximants seems to link each complex pair together across the positive and the negative real axes, respectively, and perhaps also to the point at infinity. In order to produce approximants which can be defined as the analytic continuation along the real (physical) axis, we use the method of Hunter and Baker[17] (see also Baker[18] and Baker *et al.*[19]). Therefore we have chosen to define an approximant to a formal power series $f(z)=\sum f_j z^j$ by first defining the polynomials P_L, Q_M, and R_N of degrees L, M, and N, respectively, by

$$Q_M(z)\,f'(z)+P_L(z)\,f(z)-R_N(z)=O(z^{L+M+N+2}) \tag{4.1}$$

The approximant (called an integral approximant) will be the solution of

$$Q_M(z)\;y'(z)+P_L(z)\;y(z)-R_N(z)=0 \tag{4.2}$$

and we will denote it by $y=[N/L;\,M]$ in the modern notation. The advantage of this approach is that it can be integrated along the real axis directly between the pair of singularities and so has the possibility of reproducing the physical branch of the function. We have formed the $[n/5;4]$ for $n=1,\ldots,10$. The results are displayed in Table II for the coefficient u of K^2 in (2.3). From (2.3) the bounds on u in $\ln\xi$ are

$$2.1884396\cdots\leqslant u\leqslant 8.753758\ldots \tag{4.3}$$

for reference purposes.

It will be observed that the series of values is just beginning to settle down into the allowed range. This result is in accord with the Hunter and

Table I. ln χ

K	Ising model	Border model	Gaussian model
0.0	0.0	0.0	0.0
0.1	0.2	0.2132265	0.2231436
0.2	0.4	0.4590539	0.5108256
0.3	0.6	0.7490163	0.9152907
0.4	0.8	1.0988312	1.6094379
0.5	1.0	1.529626	∞
0.6	1.2	2.06758	
0.7	1.4	2.738	
0.8	1.6	3.54	
0.9	1.8	5.	

Table II. Estimates of the Asymptotic u in $\ln \chi$

n	u
1	-0.06172
2	-0.41710
3	-0.04287
4	2.26597
5	-5.38088
6	5.273292
7	13.76704
8	11.87216
9	12.57838
10	8.72926

Baker[17] dictum that it requires about six coefficients to represent the behavior at each singularity. In this case there are the four singularities mentioned above (which is why we choose $M=4$) plus the one at infinity [the solution has the asymptotic form (2.3) when $L=M+1$], so we would expect to need about 30 coefficients instead of the 21 we have currently available. This situation suggests the value of further work, which we believe is feasible in one dimension, to extend the length of the known series.

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